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A PACKING GENERATION SCHEME FOR THE GRANULAR ASSEMBLIES WITH PLANAR ELLIPTICAL PARTICLES

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SUMMARY

In this paper, a new generator algorithm and a computer program PG2D is introduced for 2D numerical simulation of packing configuration in a granular material composed of elliptical particles of different a/b aspect ratios. Each elliptical particle is approximated by four connected arcs. The centre co-ordinates and radius of each arc and co-ordinates of connecting points can be determined from the formulae derived by entering the major axis length, $2a$, and the eccentricity. The domain to be filled with particles can be a polygon of any shape. Given the size of the packing domain, geometrical information and numbers of particles to be generated, the packing location of each particle and the co-ordinates of contact points along with contact normal rose diagram can be generated as outputs.

Simulation results show that this new algorithm can provide quite a reasonable packing model in accordance with the initial input required for the analysis of the mechanics of granular material. This generation scheme has the potential to cover packing generation and behaviour analysis of 3D sphere or ellipsoidal shaped granular materials. © 1997 by John Wiley & Sons, Ltd.

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1. INTRODUCTION

The study of mechanical behaviour of granular media has attracted considerable scientific attention over the past ten years. This, certainly, does not come as a surprise given the important role granular media play in various applications such as powder metallurgy, soil-liquefaction, under explosions, impact, blasting and fracture-failure behaviour, etc. A number of prominent researchers have made significant contributions to the mechanics of granular materials. The studies performed thus far (e.g. References 1–10) are either experimental in nature or address the aspect of formulating macroscopic constitutive relationships that take into consideration the misconstrue. Yet others deal with the development of appropriate numerical models using finite element, finite difference or discrete element schemes.

The numerical technique was first applied to geomechanics by Cundall[11] to study the dynamic behaviour of rock masses. If later, the Distinct Element Method (DEM) originally introduced

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by Cundall and Strack[12] has become a standard numerical simulation technique for research on the micromechanics of granular materials or on mechanical system of discrete natures (e.g. References 13–17).

Numerical simulation of granular assemblies can give insight into micromechanical behaviour and facilitate development of micromechanics-based constitutive model of discrete materials. Also, unlike physical and analytical modelling, numerical simulation can provide essentially any desired piece of information (stresses, strain and detailed micromechanical statistics and spatial distribution of fabric parameters) at any time step throughout loading. For any numerical simulation, the packing of the granular assembly is required as the essential input configuration. The development of an automatic packing generator can save the researcher's energy in entering the geometrical data of the granular congregated medium. In addition, the generated packing can be made more similar to the natural material giving a better analysis.

During the past 30 years or so, packing of disks and spheres of equal radii in 2D and 3D has been studied extensively by both experimental and theoretical means, in part because they serve as useful models for a variety of amorphous materials such as powder, molecular fluids and glasses. Reference 18, etc. has pointed out that there are three models which are commonly employed for packings of disks and spheres. These models are dense ordered packing, dense random packing, and loose random packing. It is not difficult to realize that the contact detection algorithm is the core feature of all the packing generation techniques and mechanical behaviour analysis for discrete system. Many researchers are still developing associated techniques and algorithms to improve analysis quality and accelerate analysing speed. (e.g. References 17, 19–21). But it is worth pointing out that the research on packing thus far have mostly focused on developing disk and sphere-based numerical codes for simplicity. BALL,[22] TRUBAL,[11] CONBAL,[23] GLUE[24] and DISC[25] are some of the available codes for circular particles.[26]

But, for rounded particle system, it has been hypothesized and studied that particle rolling can dominate as a deformational mechanism, resulting in very low aggregate shear resistance.[27,28] For less rounded particles, more interlocking occurs, inhibiting this rolling tendency. This type of effect can explain the lower prediction of peak friction angle and peak dilation rate of real sand by disk shaped particle assemblies.[24] To approximate the geometrical shape of real material, polygonal particle model is another candidate. But this type of approximation has the following drawbacks: (1) presents difficulties at the vertices for contact analysis, (2) no unique outward normal at each vertex, (3) requires large number of sides and corners to approximate rounded or subangular granules. As a result, it is natural to consider using elliptical particle to model the real material based on the following conditions: (1) ellipse having unique outward normal and no singularity at every point along its surface, (2) introducing only two additional parameters (eccentricity and orientation) in the description of particles rather than in disk, (3) may easily to be extended to 3D. Owing to the difficulties and complexity encountered in the ellipse-based discrete element model, so far, only a few researchers have worked continuously on this topic (e.g. References 25, 26, 29–32).

Bathurst and Rothenburg[29] simulate the ellipse particle packing, starting from the loose random packing of disks and then going on to adjust the eccentricity e of each particle to form the final assembly. This algorithm will cause non-uniform distribution of particle aspects (mean radius and eccentricity). In the real applications, sometime, we require the packing to be formed by certain fixed size particles and this type of packing generation algorithm is too restricted to satisfy the requirements. While on Ting's[31] and Rothenburg and Bathurst's[29] packing generation and analytical methods, it is required to perform the contact detection of ellipse particles by solving geometric equations of ellipse. A lot of effort has been made to avoid error in solving the contact point location, especially when two ellipses have small interparticle penetration. It is

accepted that using ellipse to simulate the behaviours of real granular material is a more general description than using disk shaped particles. But, it has to be kept in mind that the main purpose of the type of modelling by using well-behaved geometrical description function is to prevent the rolling effect into the analysis. From the experience of previous researchers on planar elliptical particles, using continuous ellipse function could become a more complicated problem when it is extended to deal with 3D ellipsoidal shaped particle assemblies. Therefore, in this paper we propose a new method which can approximate the ellipse in a more simple and flexible manner and demonstrate the capability of this algorithm when applied to the packing generation of elliptical shaped particles.

2. APPROXIMATION OF AN ELLIPSE

The main goal of original research is to find a simple way to approximate an ellipse that can also be easily applied to the analysis of granular mechanics. Then the question raised is that what kind of mathematical function closely approximate the shape of ellipse. As we know in the drawing theory, an ellipse with given major and minor axes $2a$ and $2b$, respectively, can be approximated by using a compass. This reminds us that an ellipse can be approximated by arcs connected piecewise. Since a circle is of simple mathematical form, it has been used in the analysis of granular medium for a certain time. Besides, the adoption of circles in the analysis of ellipse can make the contact detection process convenient. It can also be applied to the experience, accumulated in previous research, on disk shaped particle assemblies. As shown in Figure 1, following the procedures of drawing theory, the associated formulae for the 4-arc approximation of an ellipse centred at origin with major and minor axes $2a$ and $2b$ can be derived and briefly summarized as follows:

$$G\colon \left[\frac{(a^2-b^2)c+(a^2+b^2)d}{2ac},0\right] \tag{1}$$

$$I\colon \left[0,\ -\frac{(a^2-b^2)c+(a^2+b^2)d}{2bc}\right] \tag{2}$$

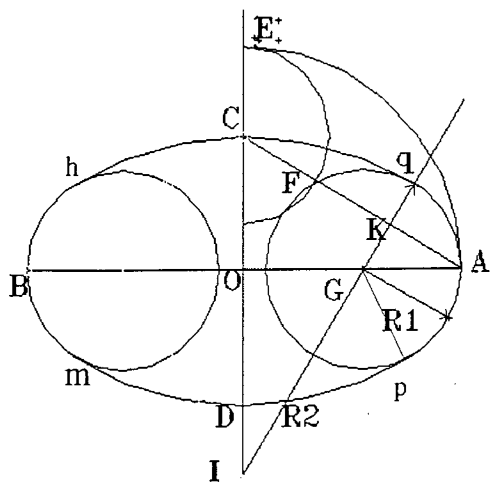

Figure 1. Approximation of an ellipse by four connected arcs

$$R_1 = \frac{c(c-d)}{2a} = \overline{GA} \tag{3}$$

$$R_2 = \frac{c(c+d)}{2a} = \overline{lq} \tag{4}$$

$$q = \left(\frac{(a^2 - b^2) + d(c-b) + bc}{2a}, \frac{c-d}{2} \right) \tag{5}$$

where

$$c = \sqrt{a^2 + b^2}, \quad d = a - b \tag{6}$$

It is found that the angle of ∠qGp is 113° and the angle of ∠qIh is 67°. These properties are useful in developing the packing generation algorithm. From this newly proposed approximation method of ellipse, it can be seen that each elliptical particle is described by four connected arcs with unique outward normal and no singularity at every point on the boundary, even at those connected points of arcs. Based on equations (1)–(6), the co-ordinates of the end points and centre of radius of each arc, for any ellipse translation from origin with a rotation with respect to horizontal axis, can be easily obtained by superimposing the rigid body node. In the development of simulation code, each ellipse has four entities stored as the basic data. In each entity, the co-ordinates of the start point, end point, radius centre of the arc and the arc radius are recorded. Those arc entities are ordered by the sequence of each arc formed through the contact detection algorithm.

3. CONTACT DETECTION

In order to satisfy the no-interpretation constraint and to model the loading transfer mechanism of particle packing, contact detection is an essential item in al the analyses of granular mechanics. In this paper, all the ellipse are approximated by 4 arcs. Therefore, the intersection of arcs is the basic element in the present analysis model. Distinct from the detection of the intersection of circles, there are some conditions for the detection of intersections between arcs and intersection between line and arc. In Figure 2, it can be seen that even if the circle on which an arc is located is

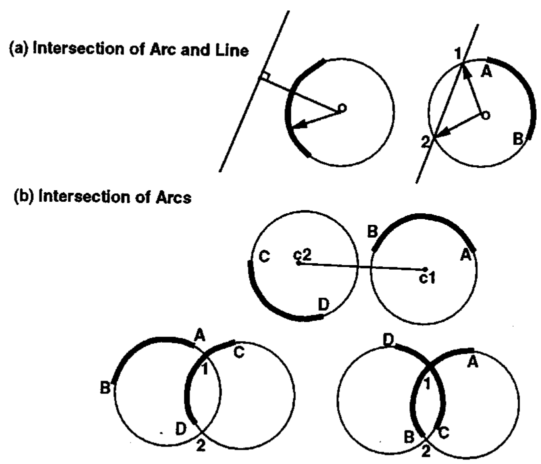

Figure 2. The intersection conditions of arc with line and another arc

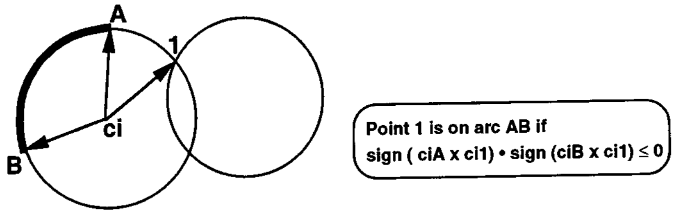

Figure 3. The criterion to verify whether a point is on an arc or not

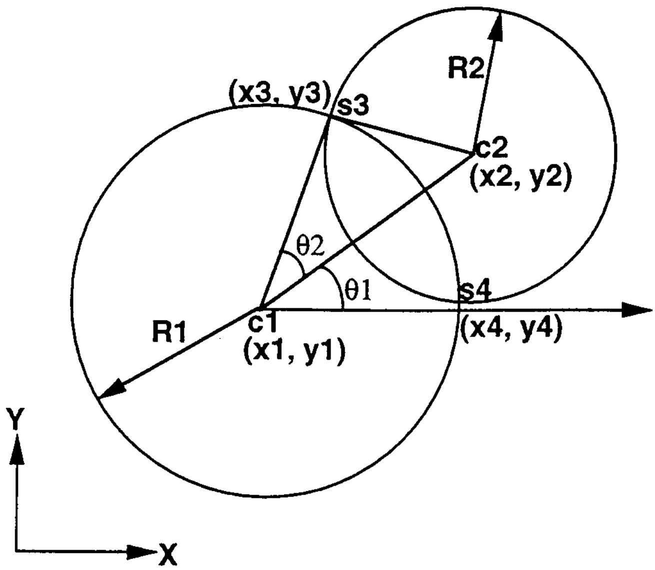

Figure 4. Determination of the coordinates of intersection points between two circles

intesected with a line or another circle, the arc itself may not have an intersection point on it. The criterion to judge the intersection of two arcs is to check whether the intersection point is located both on these two arcs or not. While the criterion to verify whether a point located on an arc or not is (see Figure 3):

$$\operatorname{sign}(\mathbf{ciA} \times \mathbf{ci1}) \cdot \operatorname{sign}(\mathbf{ciB} \times \mathbf{ci1}) \leqslant 0, \text{ then point 1 is on arc AB} \tag{7}$$

The co-ordinates of the intersection points of two circles (see Figure 4) of radius R_1 and R_2 centred at (x_1, y_1) and (x_2, y_2), respectively, can be derived and expressed as follows:

$$x_3 = x_1 + \frac{(x_2 - x_1)(R_1^2 - R_2^2 + L^2) - 4(y_2 - y_1)\sqrt{S(S - R_1)(S - R_2)(S - L)}}{2L^2} \tag{8}$$

$$y_3 = y_1 + \frac{(y_2 - y_1)(R_1^2 - R_2^2 + L^2) + 4(x_2 - x_1)\sqrt{S(S - R_1)(S - R_2)(S - L)}}{2L^2} \tag{9}$$

$$x_4 = x_1 + \frac{(x_2 - x_1)(R_1^2 - R_2^2 + L^2) + 4(y_2 - y_1)\sqrt{S(S - R_1)(S - R_2)(S - L)}}{2L^2} \tag{10}$$

$$y_4 = y_1 + \frac{(y_2 - y_1)(R_1^2 - R_2^2 + L^2) - 4(x_2 - x_1)\sqrt{S(S - R_1)(S - R_2)(S - L)}}{2L^2} \tag{11}$$

where

$$L = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \qquad S = \frac{(R_1 + R_2 + L)}{2} \tag{12}$$

Compared with the contact detection procedure of using general elliptical equation, this newly proposed method is much easier and it is not required to solve the complicated fourth-order equation. Along with this contact detection strategy, the 4-arc ellipse approximation method proposed in this paper can be adopted in the analysis model developed by Rothenburg and Bathurst[29] and Ting[31] without changing their main structures too much.

4. PACKING GENERATION ALGORITHM

The packing generation method is a technique with many degrees of freedom during code development. However, the essential constraint is not to violate the deformation compatibility condition among particles and the goal is to simulate a reasonable configuration of the inside of real materials. Therefore, different researchers have their own philosophy and the corresponding packing generation algorithms. The packing generation procedure of present paper can be explained by Figure 5, where Ω is the domain to be filled with certain types of elliptical particles. Particles enter the domain one after the other at an admissible position. The ratios among these types of particles can be designed as an input. Monte Carlo random choosing method is applied to decide the type of incoming particle at each step. For the selected particle the Monte Carlo method is applied again to decide which arc of the 4-arc approximated ellipse is the first incoming one to the generation process. After this, contact detection algorithm begins the location searching process for this incoming arc with the existing ones. Since there are many possible positions located for incoming particle, in the present model, it uses the previously generated particle as the reference for conducting the location searching process. A searching window is created from the latest formed arc of the referenced particle. As shown in Figure 6, a searching window CHEF is opened from the arc AB with length of vectors **o1C** and **o1D** equal to two times of the bigger radius of the incoming particle. The co-ordinates of point F are determined by the

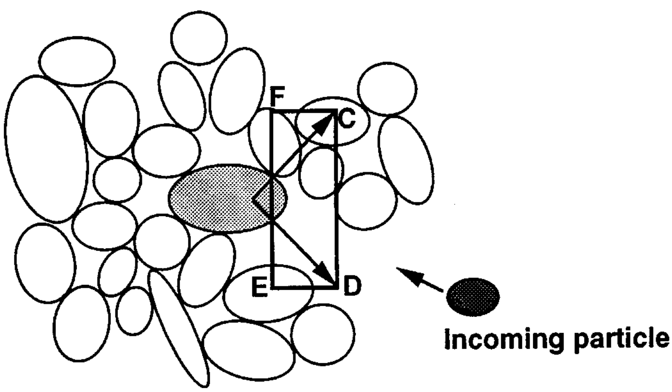

Figure 5. The schematic diagram of the packing generation procedure for elliptical particles

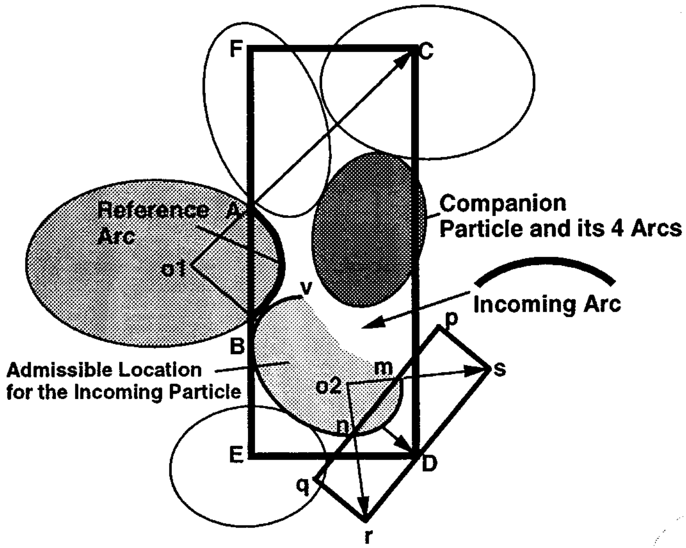

Figure 6. The contact searching windows of elliptical particles packing

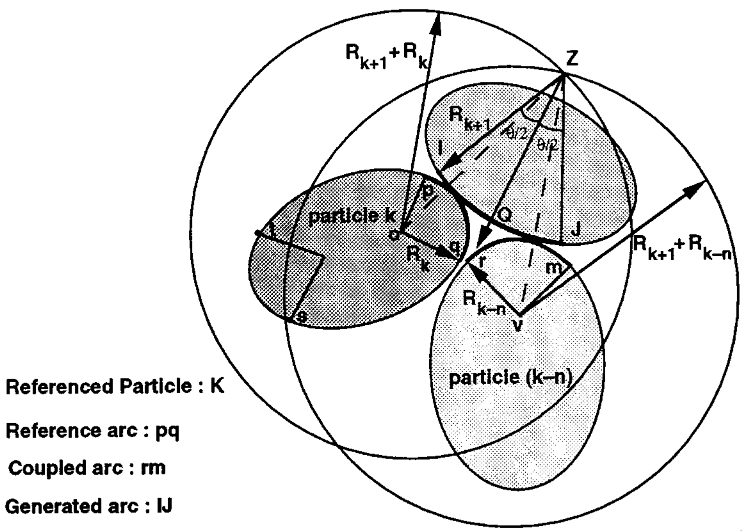

Figure 7. The determination of the location of first arc of incoming particle

unit direction of vector **BA** and the projection of vector **AC** on vector **AB**. Once the searching window is opened, all the data for the arcs, already located inside this region, are read in for contact detection check.

In order to let each particle have more contact points with other particle and to prevent overlapping conditions among particles, an intersecting circles method is developed to assist the determination of the location of arc centre. In this method, a reference circle of radius equal to the radius of reference arc plus the radius of incoming arc is formed. Another companion circle is formed with a radius equal to the radius of an arc inside the searching window plus the radius of the incoming arc (see Figure 7). All the intersection points of the referenced circle, with the

companion circles created from all the arcs inside the opened window, form a set. Every element in this set is a possible location for the centre of the incoming arc and can prevent the occurrence of overlapping (penetrating) conditions among the particles. Coordinates of these points can be determined by using equations (8)–(12). In Figure 7, point Z is the radius centre for the candidate arc and the vector **ZQ** of length equivalent to the arc radius divides the angle $\angle$oZv evenly. The co-ordinates for the end points I, J of the first incoming arc are obtained from vector **ZQ** by increasing or decreasing the augmented angle $\theta/2$. Here θ is the corresponding angle (113° or 67°) of the arc in the approximate ellipse.

After the first arc of an ellipse is formed, the remaining three arcs are generated clockwise. At any step when an arc is formed, contact detection proceeds to check the intersection conditions of this arc with the ones existing and with the boundary of the packing domain. For any violation of the non-penetrating criterion, the candidacy of the arc to be a part of elliptical particle is canceled. In the present model, a smaller window-like 'pqrs' in Figure 6 is created, from an arc, to conduct contact detection in all the arcs contained in the windowed region. In any referenced arc, there are many possible positions for an incoming particle to be located inside the windowed region. The algorithm will choose the one having the max. number of contact points with other particles. If there is no allowable position, the next arc will be assigned as the referenced arc and repeat the whole packing generation process presented as above. The early generated arc is early chosen to be a referenced arc. The whole packing generation process is terminated when all the admissible arcs have been served as the referenced one.

5. IMPLEMENTATION AND EXAMPLES

Based on the theories developed in the previous sections, a computer simulation code PG2D based on the AutoCAD-Release 12 version software was programmed. This code is written by

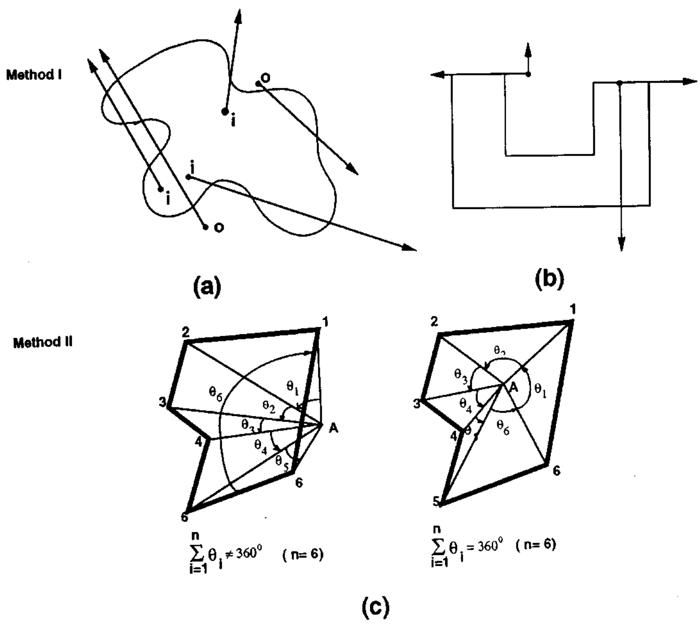

Figure 8. A topology method to evaluate an inside point of a closed region

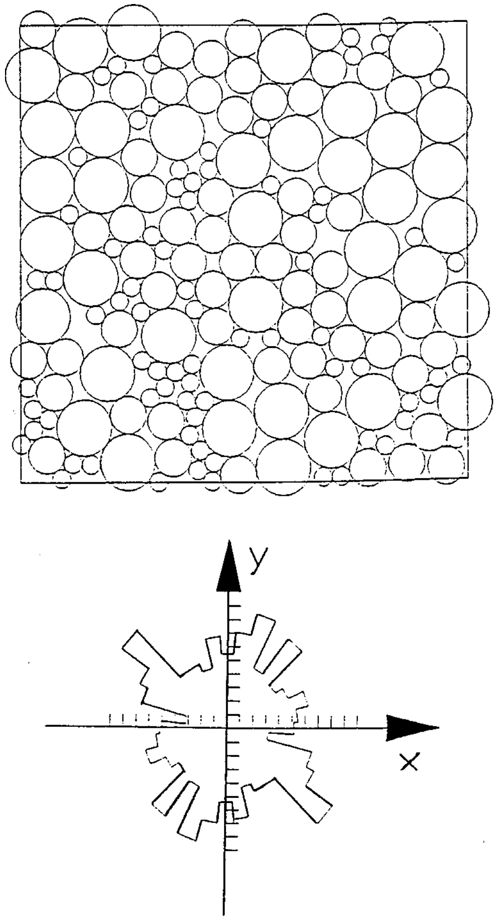

Figure 9. Packing simulation of circular disks of different sizes and the rose diagram of contact normals

AutoLISP which is a built-in language of AutoCAD. The open window function in the AutoCAD computer software is linked with the automatic packing generation code PG2D. This open window function provides a convenient working environment during code development. To ensure the arc generated is completely inside the desired packing domain W, a topological method is applied. In topology theory, shooting a line in arbitrary direction from any point on the arc, if the number of the intersection points with the domain boundary is odd then the point is inside the region, otherwise it is outside the boundary (see Figure 8(a)). Although, there are other methods that use the summation of the successional angles between the checked point with the vertices of the polygon, to evaluate whether a point is inside a closed curve or not (see Figure 8(b)). But under the working environment of AutoCAD and avoiding numerical error, the first topological method was used to proceed the check. In the present code PG2D, two orthogonal lines are shot from each inspected point to avoid some special cases like the one arising when only a line lying on a piece of the boundary is shot to cause failure of check (see Figure 8(c)).

This four arc-based elliptical particle packing generation program is validated by the following examples. Figure 9 shows the packing simulation of circular disks of three different sizes and its

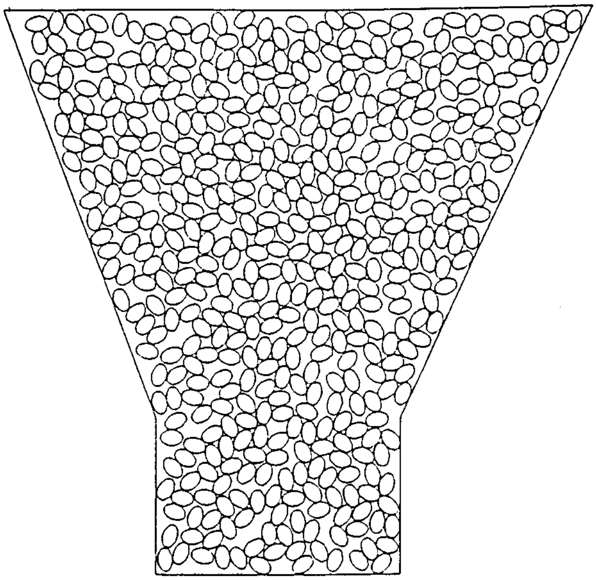

Figure 10. Packing simulation of elliptical particle of uniform size

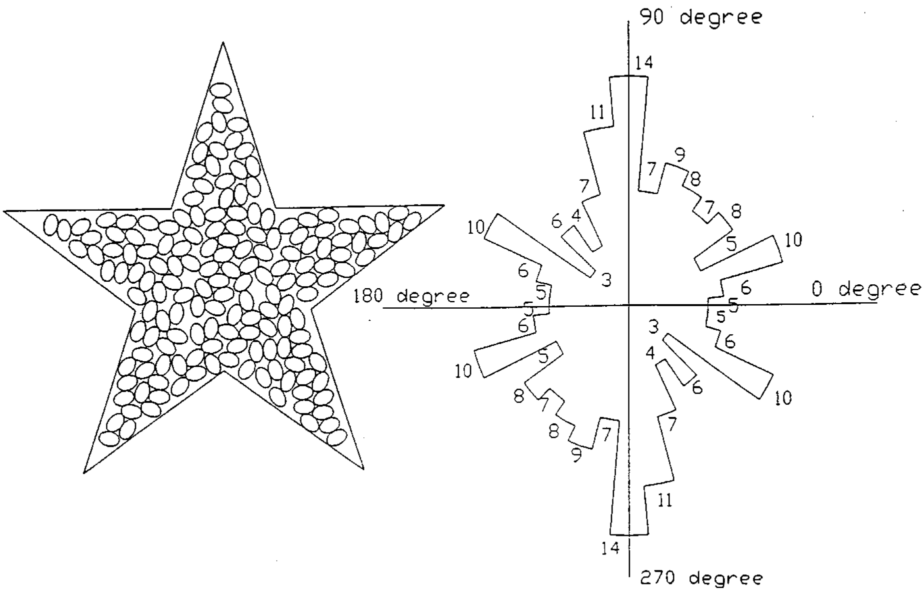

Figure 11. Packing simulation of elliptical particles in a star shaped domain and the rose diagram of contact normals

rose diagram of contact normal by setting the aspect ratio a/b of ellipse equal to one. Figure 10 demonstrates the packing of elliptical particles of uniform size in a silo shaped domain. It is believed that if there are more different types of elliptical particles the packing will be denser. Figure 11 shows the capability of the simulation code to do packing generation inside an

arbitrarily defined polygon and process some statistical data of the packing configuration. All the geometry data for every elliptical particle are stored and can be recalled as the initial configuration for further numerical analysis.

6. CONCLUSIONS

A four arcs ellipse approximation method was presented in this paper. There is a unique outward normal, and there is no singularity at every point on the boundary of this 4-arc approximated ellipse. This type of approximation can, efficiently, maintain the features of ellipse and provide a simplified mathematical form to assistant analysis in performing contact analysis of elliptical shaped particle assemblies. Based on this character, either, a new formulation of the mechanical behaviour analysis of elliptical assemblies can be achieved or it can be adopted in existing models developed by other researchers as alternative contact detection schemes. A preliminary, automatically packing generation algorithm of planar elliptical particles has been demonstrated by applying the 4-arc ellipse approximation method. Packings consisting of particles with different size and aspect ratios can be generated within a given polygon of arbitrary shape. The generated packing configuration can be used as the initial state for the numerical analysis of granular material behaviour. It is believed that this method can be easily extended to the packing generation of 3D sphere or ellipsoidal granular materials.

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NOTATION

a, b	major and minor axes for ellipse
e	eccentricity of ellipse
L	distance between two centres of two circles
R_1	radius of the approximation arc centred at the major axis
R_2	radius of the approximation arc centred at the minor axis
x, y	Cartesian co-ordinate variables
x_1, y_1, x_2, y_2	co-ordinates of the centre of two circles, respectively
x_3, y_3, x_4, y_4	co-ordinates of the intersection points of two circles

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